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TOTALLY GEODESIC FOLIATIONS

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0. Introduction

In recent years the study of foliations has become one of the most elegant and fruitful areas of research in mathematics [7]. Foliations are also of fundamental importance in differential geometry; however, the study of the geometric aspects of foliations *per se* has received considerably less attention. This paper considers a natural geometric setting for these topological structures, where each leaf is assumed to be a totally geodesic submanifold of the ambient space. Precisely, we have

Definition (0.1). Let M^n be a smooth manifold. A *codimension-k foliation* $\mathcal{F}$ of M is a decomposition of M into a union of disjoint connected codimension-k submanifolds $M = \bigcup_{L \in \mathcal{F}} L$, called the *leaves* of the foliation, such that for each $m \in M$, there is a neighborhood U of M and a smooth submersion $f_U: U \to \mathbf{R}^k$ with $f_U^{-1}(x)$ a leaf of $\mathcal{F}|_U$, the restriction of the foliation to U, for each $x \in \mathbf{R}^k$.

Definition (0.2). Let M be a Riemannian manifold, and let $\mathcal{F}$ be a codimension-k foliation on M. $\mathcal{F}$ is *totally geodesic* if each leaf L is a totally geodesic submanifold of M; that is, any geodesic tangent to L at some point must lie within L.

The two basic questions in this realm are:

Q_1: *Given a Riemannian manifold M, does it admit a totally geodesic foliation of a given codimension?*

Q_2: *Given a foliation $\mathcal{F}$ on a manifold M, is there a Riemannian metric on M so that $\mathcal{F}$ is totally geodesic; that is, is $\mathcal{F}$ geodesible?*

If the dimension of $\mathcal{F}$ is one, H. Gluck has recently made significant progress on these questions [1]. In particular, Gluck has shown that any closed 3-manifold has a geodesic flow, and has characterized those flows on 2-manifolds which are geodesible. However, in [4] the authors have shown that the codimension-one case is considerably more restrictive. In fact, given

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any compact manifold M with $\chi(M) = 0$, there are codimension-one foliations on M which are not geodesible. Also, any compact 3-manifold with a codimension-one totally geodesic foliation must have infinite fundamental group.

In the present article we classify all compact manifolds M admitting a codimension-one totally geodesic foliation with at least one compact leaf. If the foliation is transversely orientable, M is shown to fiber over a circle, with fiber the compact leaf (any two compact leaves are necessarily isometric). Geometric restrictions on the manifold are also obtained; in particular, the leaves of any totally geodesic foliation are shown to be locally isometric in any codimension. In codimension one stronger relations are shown between the isometry group of the ambient manifold and that of the leaves; in particular, if the leaves are compact the space of Killing vector fields decomposes.

Remark. R. Hermann, B. Rienhart, et al. have studied an alternate geometrization of foliations, called *bundle-like* or *fiber-like metrics*. M has a *fiber-like metric compatible with* $\mathcal{F}$ if the local submersions defining $\mathcal{F}$ may be chosen to be Riemannian submersions for a suitable choice of metrics on the images [3]. Equivalently, the leaves are assumed locally to remain a constant distance apart. Relations between this notion and totally geodesic foliations are developéd below.

1. The General situation

Throughout this paper $\mathcal{F}$ will denote a foliation of codimension k in a Riemannian manifold M^n of dimension n with Riemannian metric $\langle\, , \rangle$. References to dimension will be dropped where unnecessary. $\mathcal{F}$ will also denote the associated integrable distribution on $T_*(M)$, $\mathcal{H} = \mathcal{F}^\perp$ will be called the *horizontal* distribution; when needed, $\mathcal{F}$ will be called the *vertical* distribution. Vector fields in $\mathcal{H}$ or $\mathcal{F}$, respectively, will be referred to as *horizontal* or *vertical* as well. The symbols $\mathcal{H}$ and $\mathcal{F}$ will also refer to the orthogonal projections onto the indicated distributions.

Definition 1.1. For $E, F \in \mathcal{X}(M)$, define tensors $A_E F$, $T_E F$, of type (1,2) by

$$A_E F = \mathcal{F}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{F}F, \quad T_E F = \mathcal{F}\nabla_{\mathcal{F}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{F}E}\mathcal{F}E,$$

where ∇ is the Riemannian covariant derivative operator on M.

Remark. As in [9] it is easily seen that A and T are tensors. Many of their properties are similar to those described for the analogous tensors in [9], thus a complete description is not given here.

Lemma (1.2).

(1) $T_V W = T_W V$ *for* V, W *vertical (i.e., tangent to* $\mathcal{F}$*), and* $T = 0$ *if and only if* $\mathcal{F}$ *is totally geodesic.*

(2) $A_X Y = A_Y X$ *for* X, Y *horizontal (tangent to* $\mathcal{H}$*) if and only if* $\mathcal{H}$ *is integrable.*

(3) $A_X Y = -A_Y X$ *for* X, Y *horizontal if and only if the metric on* M *is fiber-like, compatible with* $\mathcal{F}$.

(4) $A = 0$ *if and only if* $\mathcal{H}$ *is totally geodesic.*

Proof. All except (3) are in [9]. If the metric is fiber-like, O'Neill [9] verifies (3) by the definition of fiber-like metrics. Now, if $A_X Y = -A_Y X$ for horizontal X, Y, then $A_X Y = \frac{1}{2}\mathcal{F}[X, Y]$. Let f_U: $U \to \mathbf{R}^k$ be a local submersion defining $\mathcal{F}|_U$. To find a metric on $\mathbf{R}^k$ making f_U a Riemannian submersion, let X, Y be horizontal vector fields on U which are f_U-related to $\bar{X}$, $\bar{Y}$ on $\mathbf{R}^k$. If $\langle X, Y\rangle_{|f^{-1}(x)}$ is constant, defining $\langle \bar{X}, \bar{Y}\rangle_x = \langle X, Y\rangle_{f^{-1}(x)}$ will yield the necessary metric. If V is any vertical vector field

$$V\langle X, Y\rangle = \langle \nabla_V X, Y\rangle + \langle X, \nabla_V Y\rangle$$
$$= \langle [V, X] - \nabla_X V, Y\rangle + \langle X, [V, Y] - \nabla_Y V\rangle.$$

As V is f_U-related to 0, and X is f_U-related to $\bar{X}$, $[V, X]$ is vertical, being f_U-related to 0. Thus

$$V\langle X, Y\rangle = \langle V, \nabla_X Y\rangle + \langle V, \nabla_Y X\rangle = 0,$$

as $\mathcal{F}\nabla_X Y = \frac{1}{2}\mathcal{F}[X, Y]$. q.e.d.

If $\mathcal{F} = \ker(\pi_*)$, where π: $M \to B$ is a global Riemannian submersion, O'Neill shows that the tensors A and T determine the geometry of the total space given the geometry of the fibers and the base. In our more general setting a similar result holds, though the determination is considerably less precise. This result was originally due to Kashiwabara [5], but the proof given here is quite different and will prove useful in the sequel.

Proposition (1.3). *If* A *and* T *are both* 0, M *is locally isometric to a Riemannian product.*

This proposition is clearly established by applying the following result twice, once to $\mathcal{F}$ and then to $\mathcal{H}$, yielding the required local trivialization.

Proposition (1.4). *Assume that* $\mathcal{F}$ *is totally geodesic. Given* $m \in M$*, there is a neighborhood* V *of* m *so that, if* L_1, L_2 *are leaves of* $\mathcal{F}|_V$*, then* L_1 *and* L_2 *are isometric.*

Proof. Choose a local trivializing neighborhood U of m with f_U: $U \to \mathbf{R}^k$ a local submersion defining $\mathcal{F}_{|U}$. Set $x_0 = f_U(m)$, and let $L_0 = f_U^{-1}(x_0)$. For $x \in \mathbf{R}^k$ in a small enough neighborhood of x_0, choose a regular curve $\bar{\gamma}$ from x_0 to x, $\bar{\gamma}(0) = x_0$, $\gamma(1) = x$. By an argument standard in the geometry of connections, for any $p \in L_0$ close enough to m, there is a unique horizontal lift γ of $\bar{\gamma}$ such that $\gamma(0) = p$. Let $\phi(p) = \gamma(1)$. If $L_1 = f_U^{-1}(x)$, ϕ maps a neighborhood of m in L_0 onto a neighborhood of L_1. By suitably shrinking U to V the proposition will be verified once ϕ is shown to be an isometry.

If $p, q \in L_0$ are close enough, there is a unique geodesic α in V from p to q. As $\mathfrak{F}$ is totally geodesic, α lies within L_0. Assume α is parametrized by arc length, with $\alpha(0) = p$, $\alpha(l) = q$, where $l = \operatorname{dist}(p, q)$. For each t, if γ_p and γ_q are the horizontal lifts of $\bar{\gamma}$ beginning at p and q, respectively, there is a unique geodesic α_t from $\gamma_p(t)$ to $\gamma_q(t)$. Let l_t be the length of α_t, so that $\alpha_t(0) = \gamma_p(t)$, $\alpha_t(l_t) = \gamma_q(t)$. As $\mathfrak{F}$ is totally geodesic, $\alpha_t \subset f_U^{-1}(\bar{\gamma}(t)) \in \mathfrak{F}_{|U}$.

It suffices to show l_t is a constant $(= l)$, for then $\operatorname{dist}(\phi(p), \phi(q)) = \operatorname{dist}(\gamma_p(1), \gamma_p(0)) = l_1 = l$. Define an embedding Σ of $G = \{(s, t) | 0 \leqslant s \leqslant l_t, 0 \leqslant t \leqslant 1\}$ into M by $\Sigma(s, t) = \alpha_t(s)$. The coordinate system $\Sigma^{-1}\colon \Sigma(G) \to \mathbf{R}^2$ is classically termed a *geodesic coordinate system along* γ_p of Σ. It is well-known that for such a coordinate system, the metric tensor satisfies $g_{12} = 0$; that is, $\langle d/ds\, \alpha_t(s), \Sigma_*(\partial/\partial t)\rangle = 0$. Thus the α_t are orthogonal to the curves $s =$ constant. However, γ_q is also orthogonal to α_t, thus γ_q is one of the $s =$ constant curves, implying that l_t is constant.

Remark. If $\mathfrak{H}$ is not integrable, the map ϕ will depend on the choice of path $\bar{\gamma}$. Conversely, if the isometry ϕ is determined, $\mathfrak{H}$ must be integrable, as is shown by the following

Corollary (1.5). *If the leaves admit no local isometries, then $\mathfrak{H}$ is integrable.*

Remark. The required condition is actually infinitesimal; no neighborhood of a leaf may admit a nonzero Killing field.

Proof. If $\bar{\gamma}$ in the proposition is closed, then so is γ by hypothesis. Let X, Y be horizontal vector fields on V such that X, Y are f_V-related to commuting vector fields $\bar{X}, \bar{Y}$ on $\mathbf{R}^k$. Then $[X, Y]$ is vertical, being f_V-related to zero. As $\mathfrak{F}[-, -]$ is tensorial on horizontals, it suffices to show $[X, Y] = 0$. By the standard geometric interpretation of the Lie bracket, as $[\bar{X}, \bar{Y}] = 0$, the "rectangle" of integral curves of $\bar{X}, \bar{Y}$ used (in the limit) to describe $[\bar{X}, \bar{Y}]$ are closed. By hypothesis, the lifted rectangles are also closed, hence $[X, Y] = 0$. q.e.d.

Prima facie it may appear that totally geodesic foliations are of a more restrictive nature than the fiber-like metrics of [3] and [11]. Moreover, as foliations are generalizations of fiber bundles, it may seem more natural to consider fiber-like metrics, being a straightforward generalization of Riemannian submersions. However, in the case where the orthogonal distribution is integrable (e.g. codimension one), the two notions are interchangeable.

Theorem (1.6). *If $\mathfrak{F}$ is a totally geodesic foliation on M, and $\mathcal{H}$ is integrable, then the metric on M is fiber-like, compatible with $\mathcal{H}$. Conversely, if M has a fiber-like metric compatible with a foliation $\mathfrak{F}$, and $\mathcal{H}$ is integrable, then $\mathcal{H}$ is totally geodesic.*

Proof. For the first statement, it suffices to show that, for some choice of local trivialization of $\mathcal{H}$, $f_U: U \to \mathbf{R}^{n-k}$ (k = codimension of $\mathfrak{F}$), there is a metric making f_U a Riemannian submersion. Let U be a local trivializing neighborhood of $\mathfrak{F}$, and choose $L_0 \in \mathfrak{F}_U$ ($L_0 \simeq \mathbf{R}^{n-k}$) and $V \subset U$ as in Proposition (1.4). Define $f_V: V \to \mathbf{R}^{n-k}$ by $f_V(p) = \phi(p) \in L_0 \simeq \mathbf{R}^{n-k}$ where ϕ is the local isometry constructed in that Proposition. As $f_{V^*|\ker(f_{V_*})^\perp}$ is an isometry, f_V is a Riemannian submersion.

Conversely, if the metric on M is fiber-like, compatible with $\mathfrak{F}$, let $f_U: U \to \mathbf{R}^k$ be a local Riemannian submersion for some metric on $\mathbf{R}^k$. Assuming that $\mathcal{H}$ is integrable, to show it is totally geodesic it suffices to show that $\mathfrak{F}\nabla_X Y = 0$ for vector fields X, Y in $\mathcal{H}$. By [9, Lemma 2], $\mathfrak{F}\nabla_X Y = \frac{1}{2}\mathfrak{F}[X, Y]$, which vanishes as $\mathcal{H}$ is integrable.

Corollary (1.7). *The metric on M is fiber-like compatibly with a foliation $\mathfrak{F}$ if and only if any geodesic tangent to $\mathcal{H} = \mathfrak{F}^\perp$ at one point is always tangent to $\mathcal{H}$.*

Proof. Assume any geodesic α orthogonal to $\mathfrak{F}$ at $p = \alpha(0)$ is always orthogonal to $\mathfrak{F}$. For a given local trivialization $f_U: U \to \mathbf{R}^k$ of $\mathfrak{F}$ with $p \in U$, $f_U^{-1}(f_U(\alpha))$, near p, is foliated by $\{L \cap f_U^{-1}(f_U(\alpha))\} = \mathfrak{F}_\alpha$, and $\mathcal{H}_\alpha = \mathfrak{F}_\alpha^\perp$ (in the induced metric) is totally geodesic. This follows as $\nabla_X X \in \mathcal{H}$ for any horizontal vector field X, which in turn follows from writing $X = f^i X_i$, where X_i are horizontal geodesic vector fields with $f^i(p) = \delta_{i1}$. If X is a unit vector field generating $\mathcal{H}_\alpha$, X is f_U-related to the vector field $f_{U*}(\alpha')$ along $f_U(\alpha)$, as the induced metric on $f_U^{-1}(f_U(\alpha))$ is fiber-like by Theorem (1.6). However, this must apply to any geodesic orthogonal to $\mathfrak{F}$ at p. The required metric at $f_U(p)$ on $\mathbf{R}^{n-k}$ is then defined by $|f_{U*}(\alpha'(0))| = 1$ for all such geodesics α parametrized by arclength. The converse statement appears in [11].

Suppose that the metric on M is fiber-like compatibly with $\mathfrak{F}$, and moreover is itself totally geodesic. If $\mathcal{H}$ is integrable, then applying theorem (1.6) and Proposition (1.3) we know that M will be locally a Riemannian product. If an additional assumption on the isometry groups of the leaves L of $\mathfrak{F}$ is made, a stronger conclusion follows (cf. also [2] and [11]).

Corollary (1.8). *If M has a totally geodesic foliation $\mathfrak{F}$ such that the Riemannian metric on M is fiber-like compatibly with $\mathfrak{F}$, and the leaves admit no local isometries, then M is a Riemannian product.*

Remark. Here the necessary condition is that, for no neighborhoods U, V in any leaf $L \in \mathfrak{F}$ is there a local isometry $\phi: U \to V$.

Proof. As in Corollary (1.5), the assumption on the isometry groups implies that $\mathcal{H}$ is integrable. Theorem (1.6) implies $\mathcal{H}$ is also totally geodesic, so M is locally a Riemannian product. The corollary then follows from the following lemma.

Lemma (1.9). *If $\mathcal{F}$ is totally geodesic, and $\mathcal{H}$ is integrable, then each leaf of $\mathcal{F}$ meets any leaf of $\mathcal{H}$. If the leaves of $\mathcal{F}$ have no local isometries, $L \cap H$ consists of exactly one point for each $L \in \mathcal{F}$, $H \in \mathcal{H}$.*

Proof. The second statement follows easily from the first. Assume then that $\mathcal{F}$ is totally geodesic, and let H be a leaf of $\mathcal{H}$ missing a leaf L of $\mathcal{F}$. As $\mathcal{F}^{\perp} = \mathcal{H}$, $\overline{\mathrm{L}} \cap \overline{\mathrm{H}} = \varnothing$. The closure of a leaf is clearly a union of leaves of the foliation. Thus, if $a \in \overline{L}$, $b \in \overline{H}$ are such that $\operatorname{dist}(a, b) = \operatorname{dist}(L, H) = l$, and $\alpha\colon [0, l] \to M$ is a minimal geodesic realizing that distance, with $\alpha(0) = a$, $\alpha(l) = b$, then $\alpha'(0) \perp \mathcal{F}_{\alpha(0)}$ and $\alpha'(l) \perp \mathcal{H}_{\alpha(l)}$. If $\alpha'(0)$ were not orthogonal to $\mathcal{F}$, a shorter curve between L and H could easily be constructed. But, as $\mathcal{F}$ is totally geodesic, and $\alpha'(l) \in \mathcal{F}_{\alpha(l)}$, α must be contained in a leaf of $\mathcal{F}$. This contradiction completes the proof of the lemma.

Remark. The argument presented here is a simple but very useful method which will be used repeatedly in this paper.

Pick a leaf $L_0 \in \mathcal{F}$, and define $\pi\colon M \to L_0$ by: any $m \in M$ is on exactly one $H \in \mathcal{H}$, so $\pi(m) = H \cap L_0$. Using the local isometries described in Proposition (1.4), π is readily seen to be a Riemannian submersion. [2] then implies, as $\mathcal{H}$ is totally geodesic, that $\pi\colon M \to L_0$ is a fiber bundle. If π were not a product projection, then $\pi_{|L}$, L another leaf of $\mathcal{F}$, would necessarily be a nontrivial Riemannian covering. But the action of $\pi_1(L_0)/\pi_*(\pi_1(L))$ on L yields a nontrivial isometry of L, contrary to the hypotheses. This completes the proof of Corollary (1.8). q.e.d.

In the case that the codimension of $\mathcal{F}$ is one, as $\mathcal{H}$ is automatically integrable the hypotheses in the previous corollary may be considerably weakened, and still yield interesting conclusions.

Corollary (1.10). *If $\mathcal{F}$ is a codimension-one, transversely oriented, totally geodesic foliation, and $\mathcal{H}$ is also totally geodesic, then the leaves of $\mathcal{F}$ are globally isometric.*

Remarks. This does not say that M is a Riemannian product; in particular, if M is compact the leaves of $\mathcal{F}$ need not be. Also, this corollary could be stated equivalently by assuming that $\mathcal{F}$ is totally geodesic and that the metric on M is fiber-like.

Proof. As $\mathcal{F}$ is transversely oriented, there is a unit vector field X spanning $\mathcal{H}$. As $\mathcal{H}$ is totally geodesic, X is a geodesic flow field.

Claim. *X is Killing.*

Proof. For any $E, F \in \mathfrak{X}(M)$, consider $\langle \nabla_E X, F \rangle$. $\langle \nabla_E X, F \rangle = 0$ if E or F is horizontal (i.e., in $\mathcal{H}$). If both are vertical, $\langle \nabla_E X, F \rangle = -\langle X, \nabla_E F \rangle = 0$ as $\mathcal{F}$ is totally geodesic. As the Killing condition $\langle \nabla_E X, F \rangle = -\langle \nabla_F X, E \rangle$ is tensorial in E and F, the claim follows. q.e.d.

The one-parameter group of isometries ϕ_t of X takes leaves into leaves as $\phi_{t*}(X_p) = X_{\phi_t(p)}$, thus $\phi_{t*}(\mathcal{F}) = \mathcal{F}$, and $\phi_{t|_L}: L \to L'$ is an isometry.

2. Codimension one

The purpose of this section is to classify all compact manifolds M admitting a geodesible codimension-one foliation $\mathcal{F}$, under the assumption that there is a compact leaf $L_0 \in \mathcal{F}$. By passing to a 2-fold cover, if necessary, it will be assumed throughout that $\mathcal{F}$ is transversely oriented.

Theorem (2.1). *If M is a compact Riemannian manifold with a codimension-one totally geodesic transversely oriented foliation $\mathcal{F}$ such that there is a compact leaf L_0, then M fibers over S^1, with fiber L_0.*

Remarks. (1) If $\mathcal{F}$ is not transversely oriented, M is the orbit space of a free $\mathbf{Z}_2$-action on such a fibration.

(2) The foliation $\mathcal{F}$ is *not* necessarily the trivial foliation in this fibration. For example, Gluck [1] has shown that any flow on $S^1 \times S^1$ with no Reeb components is geodesible.

The proof of Theorem (2.1) is based on the following propositions.

Proposition (2.2). *Let M be a compact Riemannian manifold with a codimension-one, transversely oriented, totally geodesic foliatioin $\mathcal{F}$. If there is a compact leaf $L_0 \in \mathcal{F}$, then $M - L_0$ is connected, and $H_1(M, \mathbf{R}) \neq 0$.*

Proof. Consider the Mayer-Vitoris sequence (using real coefficients) of the pair $M - L_0$ and V, where V is a tubular neighborhood of L_0. The sequence $\cdots \to H_1(M) \xrightarrow{\partial} H_0((M - L_0) \cap V) \to H_0(M - L_0) \oplus H_0(L_0) \to H_0(M) \to 0$ is exact. As $\mathcal{F}$ is transversely oriented, $(M - L_0) \cap V \simeq L_0 \times \mathbf{Z}_2$, so $H_1(M) \xrightarrow{\partial} \mathbf{R}^2 \to H_0(M - L_0) \oplus \mathbf{R} \to \mathbf{R} \to 0$ is exact. Either $M - L_0$ is connected, and $\partial(H_1(M)) \neq 0$, or $M - L_0$ has 2 components.

Assume now that $M - L_0$ is disconnected. Call one component of $M - L_0$ the *inside* of L_0. Choose a unit vector field X in $\mathcal{H}$ so that $X_{|L_0}$ is inward-pointing. Let γ be an integral curve of X so that $\gamma(0) = x \in L_0$.

As previously stated, $\bar{\gamma} - \gamma = \overline{\gamma(0, \infty)} - \gamma(0, \infty)$ is a union of leaves of $\mathcal{H}$. As $X_{|L_0}$ is inward-pointing, $\gamma[0, \infty)$ meets L_0 only at 0, thus $\mathrm{dist}(L_0, \bar{\gamma} - \gamma) = l > 0$. Pick $y \in \bar{\gamma} - \gamma$, $z \in L_0$ realizing this distance, and let α be a minimal unit-speed geodesic from y to z, with $\alpha(0) = y$, $\alpha(l) = z$. $\alpha'(0)$ is in $\mathcal{F}$, but $\alpha'(l)$ is in $\mathcal{H}$, contradicting the fact that $\mathcal{F}$ is totally geodesic. Thus $M - L_0$ must be connected, implying that $H_1(M) \neq 0$. q.e.d.

Let M, $\mathfrak{F}$, and L_0 be as in the previous Proposition. Construct a manifold $\tilde{M}$ with boundary $L_0 \times \mathbf{Z}_2$ by the following method: begin with the manifold-with-boundary $M - V$, V a tubular neighborhood of L_0. By the collaring theorem the two closed halves of V may be glued back onto $M - V$, including the zero-section in each half, with the Riemannian metric inherited from M. This is the manifold $\tilde{M}$; simply stated, cut M open along L_0. Note that $\mathfrak{F}$ lifts to a totally geodesic foliation $\tilde{\mathfrak{F}}$ with one additional copy of L_0. As usual, let $\tilde{\mathfrak{K}} = \tilde{\mathfrak{F}}^{\perp}$. Call $\partial\tilde{M} = L_0 \cup L_1$. Note that L_1 is identified with L_0 in such a way that $M \simeq \tilde{M}/\sim$ where $x \in L_1 \sim x \in L_0$.

The following result is an easy consequence of Lemma (1.9).

Proposition (2.3). *Each leaf $\gamma \in \tilde{\mathfrak{K}}$ meets both components of $\partial(\tilde{M})$.*

Proof of Theorem (2.1). Let X be a unit vector field generating $\tilde{\mathfrak{K}}$ in $\tilde{M}$, with $X_{|L_0}$ inward-pointing. For each $x \in L_0$, let γ_x be the integral curve of X with $\gamma_x(0) = x$. Note that $\{\gamma_x\}_{x\in L_0} = \tilde{\mathfrak{K}}$, and let t_x be that parameter value so that $\gamma_x(t_x) \in L_1$. The function $x \mapsto t_x$ is a smooth, positive function on L_0. Consider the mapping ϕ: $L_0 \times I \to \tilde{M}$ defined by $\phi(x, s) = \gamma_x(s \cdot t_x)$. As all leaves of $\tilde{\mathfrak{K}}$ are represented, ϕ is clearly surjective; as $\tilde{\mathfrak{K}}$ is a foliation ϕ must be injective. ϕ is also easily seen to be smooth and nonsingular, thus is a diffeomorphism.

Note that f: $L_0 \to L_1$ defined by $f(x) = \phi(x, 1)$ is precisely the mapping obtained by stringing together the local isometries of leaves described in Proposition (1.4), thus f is an isometry. Evidently, $M \simeq \tilde{M}/\phi(x, 1) \simeq (f(x), 0)$, considering $f(x)$ as an element of L_0 (using the identification involved in the definition of $\tilde{M}$).

Define π: $M \to S^1 \subset \mathbf{C}$ by $\pi(\phi(x, s)) = e^{2\pi si}$. π is well-defined as a mapping on M, rather than on $\tilde{M}$, by construction. That π is a submersion is straightforward; from [2] and Theorem (1.6), π is seen to be a fibering.

Corollary (2.4). *If $\mathfrak{F}$ is a codimension-one totally geodesic, transversely oriented foliation on a compact Riemannian manifold M, any two compact leaves of $\mathfrak{F}$ are isometric.*

Proof. Let L_0, L_1 be two compact leaves. Construct $\tilde{M}$ as in the theorem, using L_0. Consider ψ: $L_1 \to L_0$ defined by $\psi(y) = x$, where $x \in L_0$ is that point such that $y \in \gamma_x$. ψ is easily seen to be a locally isometric covering projection. Reversing the roles of L_0 and L_1 completes the proof.

3. Isometries and Killing fields

In this section, we consider Killing fields and isometries on a complete Riemannian manifold. Recall that a Killing field X is an infinitesimal generator of a one-parameter group of isometries. Equivalently, X satisfies the condition

$$\langle V_Y X, Z \rangle = -\langle Y, \nabla_Z X \rangle$$

for all vector fields X and Y.

Theorem (3.1). *Let M be a complete connected Riemannian manifold admitting a codimension-one totally geodesic foliation $\mathcal{F}$ by compact leaves. If a Killing field X is tangent to $\mathcal{F}$ somewhere, then it is tangent everywhere.*

Remark. The condition that the leaves be compact cannot be eliminated, even if M is assumed to be compact. For example, consider the flow on the torus $S^1 \times S^1$ obtained as follows. The graphs of $y = \arctan(x - x_0)$, for $x_0 \in \mathbf{R}$, and $y = \pm\pi/2$ foliate $\mathbf{R} \times [-\pi/2, \pi/2]$. The identifications $(n, y) \sim (m, y)$, $n, m \in \mathbf{Z}$ and $(x, -\pi/2) \sim (x, \pi/2)$ yield the required flow. A generator for this flow is given by

$$Y = \frac{\partial}{\partial x} + \frac{1}{1 + \tan^2 y}\frac{\partial}{\partial y}.$$

where $\partial/\partial x$, $\partial/\partial y$ are the standard coordinate vector fields on $S^1 \times S^1$ induced from $\mathbf{R} \times [-\pi/2, \pi/2]$. This flow has a unique compact leaf corresponding to the graph of $y = \pi/2$.

Consider a metric on the torus defined by $\|\partial/\partial y\| = \|Y\| = 1$ and $\langle \partial/\partial y, Y \rangle = 0$. Then Y is a geodesic field and $\partial/\partial x$ is a Killing field. To see that $\nabla_Y Y = 0$, first note that the 1-form dx is $dx = \langle\ , Y\rangle$. Since this form is closed, it easily follows that $\langle [Y, \partial/\partial y], Y\rangle = 0$, and hence $\langle \partial/\partial y, \nabla_Y Y\rangle = 0$. To see that $\partial/\partial x$ is Killing, observe that this flow preserves Y and $\partial/\partial y$, and hence preserves the metric. Thus there is a Killing field on the torus which is tangent to the compact leaf of the flow, but nowhere else tangent.

Also, in higher codimensions the theorem is false. The Hopf fibration $S^1 \to S^3 \to S^2$ gives a geodesic flow on the constant curvature 3-sphere. Regarding $S^3 \subset \mathbf{C} \times \mathbf{C}$, the Killing field generated by rotating one of the complex factors is tangent to exactly two circles.

Proof. First note that if Z is any vector field (generated by a one-parameter group ψ_t of diffeomorphisms) on a manifold with a codimension-one foliation by compact leaves, then $\psi_t(L)$ is tangent to the foliation at some point, for any leaf L. In our totally geodesic setting, this implies that the one-parameter group of isometries generated by any Killing field preserves the foliation.

Assume that X is somewhere tangent and transverse to L, and let p belong to the boundary of the tangency set in L. By trivializing $\mathcal{F}$ in a neighborhood of p, it is clear that X cannot take L into another leaf. Thus X is everywhere tangent to L. Now assume that X is everywhere transverse to a leaf L'. Since L and L' are compact, there is a minimizing geodesic $\alpha\colon I \to M$ between them. Suppose that $\langle \alpha'(\mathrm{a}), X \rangle$ is positive (otherwise use $-X$), and let α flow along X to generate a family α_s of geodesics, $\alpha_s(t) = \phi_s(\alpha(t))$, ϕ_s in the one-parameter group of isometries generated by$X$. For small positive s, it follows that α_s and L' intersect at $\alpha_s(t)$, for some $t < 1$ depending on s. So α_s is a shorter geodesic from L to L', a contradiction; hence X is tangent to L'. q.e.d.

As an immediate consequence, if X vanishes somewhere, then it is everywhere tangent to $\mathfrak{F}$. Let $G_{\mathfrak{F}}$ denote the identity component of the isometries of M which preserve each leaf of $\mathfrak{F}$. The associated Lie algebra consists of those Killing fields everywhere tangent to $\mathfrak{F}$. Also let G_0 denote the identity component of the isometry group of M.

Proposition (3.2). *Assuming the hypothesis of Theorem* (3.1), *then $G_{\mathfrak{F}}$ is a compact normal subgroup of G_0, and the quotient $G_0/G_{\mathfrak{F}}$ is either $\{e\}$, $\mathbf{R}$, or S^1.*

Proof. Let X (resp. Y) be a Killing field tangent (resp. transverse) to $\mathfrak{F}$ with one-parameter group of isometries ϕ_t (resp. ψ_t). To show that $G_{\mathfrak{F}}$ is a normal subgroup, it suffices to show that $[X, Y]$ is tangent to $\mathfrak{F}$. By the above remarks, ϕ_t and ψ_t preserve $\mathfrak{F}$, and it is clear that the curve $t \mapsto \psi^{-1}{}_{\sqrt{t}}\, \phi^{-1}_{\sqrt{t}}\, \psi_{\sqrt{t}}\, \phi_{\sqrt{t}}\,(p)$ lies in a leaf L for any $p \in L$. But $[X, Y]$ is the tangent vector to this curve and so $G_{\mathfrak{F}}$ is normal. As for the compactness, first observe that $G_{\mathfrak{F}}$ is closed. Let ρ_n be a sequence of isometries in $G_{\mathfrak{F}}$. If $a \in L$, any leaf, then there is a subsequence, still denoted by ρ_n, such that $\rho_n(a)$ converges. By [6, Lemma 3, p. 47], it follows that there is a subsequence of this subsequence which converges in G, and hence in $G_{\mathfrak{F}}$.

If Z is any Killing field, then by Theorem (3.1), Z can be uniquely written as a linear combination of Y and an $\mathfrak{F}$-tangent Killing field. Hence the codimension of $G_{\mathfrak{F}}$ in G is 0 or 1 depending on the existence of Y. If M is compact, then $G/G_{\mathfrak{F}}$ is either $\{e\}$ or S^1. If M is noncompact and Y exists, then no flow line α of Y intersects a leaf more than once. Otherwise, consider any two points $\alpha(t_1), \alpha(t_2) \in L$ and set $M_1 = \bigcup_{t \in [t_1, t_2]} L(\alpha(t))$, where $L(\alpha(t))$ is the leaf through $\alpha(t)$. Since Y is everywhere transverse, it follows that M_1 is open and closed. But M_1 is also compact, a contradiction. Hence the one-parameter group of isometries ψ_t is not periodic; in particular, $G/G_{\mathfrak{F}} = \mathbf{R}$. q.e.d.

We conclude with the following classification result.

Theorem (3.3). *Let M be a complete noncompact Riemannian manifold with a codimension-one transversely oriented totally geodesic foliation by compact leaves. Then all of the leaves are isometric* (*to say L*) *and M is diffeomorphic to $L \times \mathbf{R}$.*

Remark. M need not be isometric to a product; for example, put a bend in the cylinder $S^1 \times \mathbf{R}$.

Proof. As in the proofs of Theorem (2.1) and Corollary (2.4), it suffices to show that every orthogonal curve intersects each leaf exactly once. This follows as in the proof of Proposition (3.2).

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